NUMERICAL AND ASYMPTOTIC SOLUTION OF THE PROBLEM CONCERNING THE COMPLETE STABILIZATION OF A BOUNDARY LAYER

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The numerical method given in [1] is used here for calculating the temperatures of complete stabilization for a supersonic boundary layer at a flat plate with the boundary condition $\theta(0) = 0$, where θ denotes the amplitude of temperature perturbations. According to the results, the conclusion in [2] that there exist two regions of complete stabilization is wrong. The asymptotic method used in [2] is analyzed here. It is shown that two regions of complete stabilization appear to exist, because the equations used in [2] had been set up for the viscous case and, therefore, are not applicable at low surface temperatures. The results of this analysis are confirmed by direct numerical integration.

1. A numerical method has been proposed in [1] for solving problems concerning the complete stabilization of a supersonic boundary layer subject to small two-dimensional perturbations.

The calculations were performed only for the boundary condition

$$\theta'(0) = 0$$

with regard to temperature, where θ denotes the amplitude of temperature perturbations and the prime indicates a derivative with respect to the coordinate normal to the surface.

Of practical interest is the problem concerning the complete stabilization of a supersonic boundary layer also under the condition that

$$\theta(0) = 0$$

Such a boundary condition was stipulated in [2] for calculating the temperatures of complete stabilization. The results in [2] were obtained by the asymptotic method, and they may be unreliable at least within the $M \sim 2$ range of Mach numbers (see [1]). It is necessary to refine the results of [2], therefore, this will be done here by the calculation method given in [1]. The viscosity coefficient is assumed proportional to the temperature ($\mu = T$), the Prandtl number $\sigma = 0.75$, and the adiabatic constant $\gamma = 1.4$. The results of calculations for various values of the Mach number M are shown in Fig. 1: surface temperatures T_W at which complete stabilization occurs along the first neutral curve (1), and along the second neutral curve (2) (the existence of two neutral curves corresponding to intensive cooling of the surface has been demonstrated in [1]). For comparison, the results obtained in [2] by the asymptotic method are shown here with a dashed line. The comparison indicates a wide discrepancy between the results based on the numerical method and those based on the asymptotic method.

The numerical method has yielded one region of complete stabilization, which is bounded by curve 1 for $M < 3.2$ and by curve 2 for $M > 3.2$. The asymptotic method [2] has yielded two such regions. The first one is bounded by a curve (dashed line) consisting of two branches which have been labeled I and II in Fig. 1. Branch I coincides with curve 1 for $M \leq 1.4$. The second region is bounded by curve III. In order to

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TABLE 1

M	Branch I			Branch II			Branch III		
	T_w	ε	$\varepsilon T'_w/T_w$	T_w	ε	$\varepsilon T'_w/T_w$	T_w	ε	$\varepsilon T'_w/T_w$
1.1	0.586	0.017	0.009	0.22	0.033	0.20	0.166	0.032	0.37
1.25	0.798	0.050	0.011	0.394	0.094	0.16	0.304	0.091	0.29
1.43	0.929	0.097	0.014	0.554	0.18	0.14	0.427	0.17	0.26
1.67	1.031	0.170	0.021	0.742	0.25	0.10	0.552	0.24	0.22
1.8	1.055	0.220	0.030	0.872	0.28	0.08	0.62	0.29	0.21

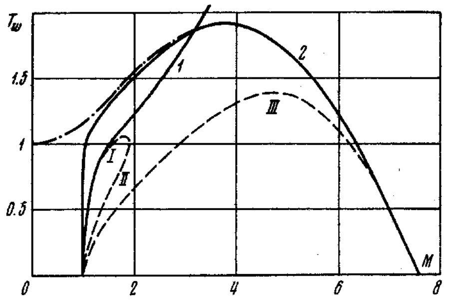

Fig. 1

understand the causes of such a discrepancy, we will thoroughly analyze the feasibility of solving the problem of complete stabilization by the asymptotic method.

2. The following system of equations has been derived in [3] for the perturbation amplitudes:

$$\frac{i(U-c)}{T} f + \frac{U'}{T}\varphi = -\frac{iP}{\gamma M^2} + \frac{\mu}{\alpha R} f'' \tag{2.1}$$

$$\frac{i(U-c)}{T}\alpha^2\varphi = -\frac{P'}{\gamma M^2} + \frac{\mu}{\alpha R}\alpha^2\varphi'' \tag{2.2}$$

$$i(U-c)rT - \frac{T'}{T}\varphi + if + \varphi' = 0 \tag{2.3}$$

$$\frac{i(U-c)}{T}\theta + \frac{T'}{T}\varphi = -(\gamma-1)(if+\varphi') + \frac{\gamma\mu}{\alpha R}\theta'' \tag{2.4}$$

$$P = rT + \theta / T \tag{2.5}$$

Here, U and T are the time-average velocity and temperature, respectively, f, $\alpha\varphi$, r, and P are perturbations of the longitudinal and the transverse velocity, of the temperature, of the density, and of the pressure; γ is the adiabatic constant, M is the Mach number, μ is the velocity coefficient, R is the Reynolds number, α is the wave number of a perturbation, and $c = c_r + ic_i$ is the phase velocity of a perturbation. In deriving system (2.1)-(2.5) it has been assumed that a perturbation is an exponential function of the longitudinal coordinate x and of time t: $\exp i\alpha(x-ct)$.

Let the solution of this system satisfy the following three boundary conditions at the surface:

$$f_w = \varphi_w = \theta_w = 0 \tag{2.6}$$

If the product αR is sufficiently large, then, the fundamental system of solutions can be made up of the solution to the inviscid equations $\{\Phi, F, \Theta\}$ and two linearly independent viscous solutions $\{\varphi_3, f_3, \theta_3\}$ and $\{\varphi_5, f_5, \theta_5\}$ [4]. Condition (2.6) will then yield the following relation at the surface [4]:

$$\frac{\Phi_w}{F_w} = \left\{\frac{\varphi_{3w}}{f_{3w}} + \frac{\varphi_{5w}}{\theta_{5w}}\left[(\gamma-1)M_c^2 - \frac{\theta_{3w}}{f_{3w}}\right] - (\gamma-1)M^2c\frac{\varphi_{3w}}{f_{3w}}\frac{f_{5w}}{\theta_{5w}}\right\}\left\{1 - i(\gamma-1)M^2c\left[\frac{U_w'}{c} - \frac{T_w'}{(\gamma-1)M^2c^2}\right]\left(\frac{\varphi_{5w}}{\theta_{5w}} - \frac{f_{5w}}{\theta_{5w}}\frac{\varphi_{3w}}{f_{3w}}\right) - \frac{f_{5w}}{\theta_{5w}}\frac{\theta_{3w}}{f_{3w}}\right\}^{-1} \tag{2.7}$$

The left-hand side of Eq. (2.7) is usually written as [5]

$$\frac{\Phi_w}{F_w} = \frac{ic}{U_w'}\frac{u+iv-1}{u+iv} \tag{2.8}$$

During complete stabilization, $\alpha = 0$ and $c = 1 - M^{-1}$, and in order to determine the critical surface temperature in (2.8), one needs only the value of $v(T_W)$ [5] which satisfies the equality

$$v(T_w) = -\pi\frac{U_w'(1-M^{-1})}{T_w}\left[\frac{T^2}{U'^3}\frac{d}{dy}\left(\frac{U'}{T}\right)\right]_{U=c=1-M^{-1}} \tag{2.9}$$

within sufficient accuracy.

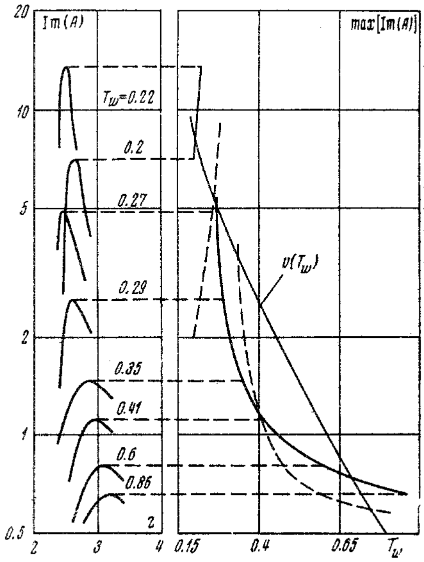

Fig. 2

The right-hand side of Eq. (2.7) depends only on the viscous solutions. If $\alpha R \gg 1$, and the following estimates are correct near the surface

$$T, T', U, U' \sim 1, \quad \frac{d}{dy} \sim \frac{1}{\varepsilon}, \quad U - c \sim 1, \quad f \sim 1 \tag{2.10}$$

$$\varphi \sim \varepsilon f, \quad \theta \sim f, \quad P \sim \varepsilon^2 f \quad (\varepsilon = (\alpha R)^{-1/2})$$

then, for the principal terms of the viscous asymptotic solutions, we can write the following system of equations [3]:

$$f''' - i\alpha R \frac{U - c}{T^2} f' = 0 \tag{2.11}$$

$$\varphi' + if = \frac{i(U - c)}{T}\theta \tag{2.12}$$

$$\theta'' - i\alpha R \frac{U - c}{T^2}\theta = 0 \tag{2.13}$$

With the viscous solutions outside the boundary layer in the form $\{\varphi_5, f_3, 0\}$ and $\{\varphi_5, O, \theta_5\}$ [4], system (2.11)-(2.13) yields

$$\theta_3 \equiv f_5 \equiv 0 \tag{2.14}$$

and expression (2.7) becomes

$$\frac{\Phi_w}{F_w} = \left\{\frac{\varphi_{3w}}{f_{3w}} + (\gamma - 1) M^2 c \frac{\varphi_{5w}}{\theta_{5w}}\right\}\left\{1 - i(\gamma - 1) M^2 c \frac{\varphi_{5w}}{\theta_{5w}}\left[\frac{U_w'}{c} - \frac{T_w'}{(\gamma - 1) M^2 c^2}\right]\right\}^{-1} \tag{2.15}$$

In the given case, i.e., when $\alpha R \gg 1$ and condition (2.10) is satisfied, the solutions to system (2.11)-(2.13) can be found analytically. Inserting these solutions into (2.15) and then, (2.15) into (2.8), we can obtain the relation which has been used in [2] for calculating the eigenvalues.

In the $M \sim 2$ range $\alpha R < 10$, and the results obtained by the asymptotic method become uncertain, because the condition $\alpha R \gg 1$ has been violated. This could explain the discrepancy between the results obtained by applying the numerical and the asymptotic method, respectively, to the problem of complete stabilization in this range of Mach numbers.

At Mach numbers close to unity the inequality $\alpha R \gg 1$ is satisfied, but the surface temperature becomes low and the condition $T \sim 1$ in (2.10) is not satisfied, which can lead to erroneous results in the asymptotic method of solution. For branch I in Fig. 1, however, the agreement between numerical and asymptotic results is very close. No solution corresponding to branches II and III has been found by the numerical method.

In order to understand the causes of this discrepancy, let us consider, for example, the derivation of Eq. (2.12). It has been derived from Eq. (2.3) under the assumption that $\varphi T'/T$ is much smaller than unity. We will now retain the estimate ε for φ, and will assume that $T \sim T_W$, whereupon, we will estimate $\varepsilon T_W'/T_W$ on the basis of the results in [2]. The quantity T_W'/T_W will be determined according to (2.9) in [2]:

$$\frac{T_w' c}{T_w U_w'} = \frac{0.4(\gamma - 1) M^2 c - 0.9(T_w - 1)c}{T_w}$$

Since it has been assumed that $\mu = T$, then, for a boundary layer at a flat plate $U_W' = 0.332/T_W$. The results of the estimates are given in Table 1. One can see here that the term $\varphi T'/T$ may be omitted for branch I. For branches II and III it is doubtful whether this term may be omitted from Eq. (2.3).* For branch I, furthermore, both ε and $\varphi T'/T$ decrease, i.e., the accuracy of the approximate calculation increases as M decreases. For branches II and III, $\varphi T'/T$ increases as ε decreases, and this yields unreliable results in the asymptotic method of solution.

*Analogous omissions have been made in deriving Eqs. (2.11) and (2.13), but they are not so obvious.

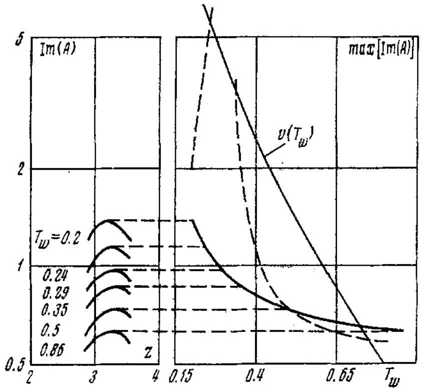

Fig. 3

Since $\varphi T'/T$ is not small for branches II and III, system (2.11)-(2.13) incorrectly describes the behavior of viscous solutions. Temperature perturbations affect one another, as can be seen from (2.1)-(2.5), and thus, condition (2.14), i.e., Eq. (2.15) used in [2], should not be satisfied.

Thus, it is erroneous in [2], first of all, that the viscous solutions were derived using system (2.11)-(2.13), which incorrectly describes the behavior of such solutions, when $\varphi T'/T$ is not small (branches II and III). Secondly, the error associated with the incorrect use of system (2.11)-(2.13) could increase considerably at low values of T_W because of the approximate integration of the equations in this system. In order to verify these statements, we have performed additional calculations.

3. It follows from (2.8) that

$$u + iv - 1 = \frac{U_w'}{ic}\frac{\Phi_w}{F_w}\left[1 - \frac{U_w'}{ic}\frac{\Phi_w}{F_w}\right]^{-1} \equiv A(z) \tag{3.1}$$

where Φ_W/F_W is determined according to (2.7). The right-hand side of equality (3.1) depends only on the viscous solutions, which can be found by direct numerical integration.

In the first series of calculations (with M = 1.2), we integrated numerically Eqs. (2.11)-(2.13). The imaginary part of the found solutions [Im (A)] was plotted as a function of z:

$$z = (cR)^{1/3}\left[\frac{3}{2}\int_0^{y_c} \frac{\sqrt{U-c}}{T}\,dy\right]^{2/3}$$

with y_c denoting the distance from point U = c along the normal to the surface. The magnitude of max [Im (A)] depends on the surface temperature. The intersections of curves $v(T_W)$ and [Im (A)] yield the temperatures of complete stabilization (Fig. 2). Such a procedure for determining the temperatures of complete stabilization was also adopted in [2], but with the difference that the viscous solutions were sought analytically. The dashed line in Fig. 2 represents the results according to the asymptotic formulas [2].

It is evident that the qualitative pattern here is the same as in [2]. The max [Im (A)] line intersects the $v(T_W)$ line, which indicates that there exist two regions of complete stabilization. The quantitative discrepancy becomes greater as T_W decreases. This discrepancy indicates that, as T_W decreases, the exact solutions to system (2.11)-(2.13) are less closely approximated by its analytical solutions.

In the second series of calculations, in order to obtain the viscous solutions, we integrated all equations of system (2.1)-(2.5). The results are given in Fig. 3. It can be seen here, that the max [Im (A)] line intersects the $v(T_W)$ line only once.

On the basis of these calculations, we can conclude that at low values of T_W, one must not use system (2.11)-(2.13) for obtaining the viscous solutions, and that the assertion in [2] concerning the existence of two regions of complete stabilization is erroneous.

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